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The trace space of the k-skeleton of the n-cube

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Abstract

We show that the space of directed paths on the k-skeleton of the n-cube is homotopy equivalent to the nerve of a certain category of flags of finite sets.

1 Introduction

In [D] Dijkstra introduced the so-called PV-model for linear concurrent programs with semaphores. The space of execution paths of such a program is a directed space. One can study it using the methods of directed algebraic topology introduced by Fajstrup, Goubault, Grandis and Raussen (see [FRG], [G]).

We will consider the following problem: Assume that s is a semaphore (that is a shared resource) of capacity k. Run n copies of the program P_sV_s in parallel $P_sV_s\|P_sV_s\|\dots\|P_sV_s$. What is the homotopy type of the associated space of execution paths?

Let us describe the space of execution paths precisely. Let $I = [0,1]$ denote the unit interval. In our setting, a directed path is a continuous curve $\gamma : I \to I^n$ with weakly increasing coordinate functions. The semaphore s is modeled by open subintervals $I_i =]a_i, b_i[\subseteq I$ for $1 \leq i \leq n$. A point $\overline{x}$ in I^n is forbidden if the set $\{i | x_i \in I_i\}$ has more than k elements. Thus the state space of the program is

$$X^n_{(k)} = \{\overline{x} \in I^n | \#\{i | x_i \in I_i\} \leq k\}$$

and we want to study the space of directed paths form $\overline{0}$ to $\overline{1}$ in this state space, or its homotopy equivalent trace space consisting of directed paths up to reparametrizations

$$\vec{P}(X^n_{(k)})(\overline{0}, \overline{1}) \simeq \vec{T}(X^n_{(k)})(\overline{0}, \overline{1}).$$

When $k = 0$, the trace space is empty. When $k \geq n$, we have a contractible trace space. So we assume that $1 \leq k \leq n-1$, since these are the nontrivial cases. Note that for $k = n-1$ the trace space is homotopy equivalent to the sphere S^{n-2} since this is a special case of [R].

2 A covering of the state space

The k-skeleton of the n-cube is the special state space where $a_i = 0$ and $b_i = 1$ for all i:

$$I^n_{(k)} = \{\overline{x} \in I^n | \#\{i | 0 < x_i < 1\} \leq k\}.$$

By a continuous deformation of each factor of I^n we obtain:

Proposition 2.1. *The inclusion $I^n_{(k)} \hookrightarrow X^n_{(k)}$ induces a homotopy equivalence of trace spaces*

$$\vec{T}(I^n_{(k)})(\overline{0}, \overline{1}) \simeq \vec{T}(X^n_{(k)})(\overline{0}, \overline{1}).$$

Proof. For $0 < a < b < 1$ we define a continuous map $H_{(a,b)} : I \times I \to I$ by

$$H_{(a,b)}(s, x) = \begin{cases} 0, & 0 \leq x \leq sa, \\ \frac{x - sa}{s(b-a)+1-s}, & sa \leq x \leq sb + 1 - s, \\ 1, & sb + 1 - s \leq x \leq 1. \end{cases}$$

If we fix $s \in I$ to any value we have a weakly increasing map of the variable x. So we can write $H_{(a,b)} : I \times \vec{I} \to \vec{I}$. Define $r_{(a,b)} : \vec{I} \to \vec{I}$ by $r_{(a,b)}(x) = H_{(a,b)}(1, x)$ and observe that $H_{(a,b)}(0, x) = x$.

We now form Cartesian products of these types of maps

$$\begin{aligned} H &: I \times \vec{I}^n \to \vec{I}^n; \quad H(s, \overline{x}) = (H_{(a_1,b_1)}(s, x_1), \ldots, H_{(a_n,b_n)}(s, x_n)), \\ r &: \vec{I}^n \to \vec{I}^n; \quad r(\overline{x}) = H(1, \overline{x}) = (r_{(a_1,b_1)}(x_1), \ldots, r_{(a_n,b_n)}(x_n)), \end{aligned}$$

so that H is a homotopy from id_{I^n} to r.

Note that r restricts to a map $\tilde{r} : X^n_{(k)} \to I^n_{(k)}$ and let $\iota : I^n_{(k)} \to X^n_{(k)}$ denote the inclusion. We also have restrictions H_1 and H_2 of the homotopy H with the following properties:

$$\begin{array}{lll} H_1 : I \times I^n_{(k)} \to I^n_{(k)}, & H_1(0, \overline{x}) = \overline{x}, & H_1(1, \overline{x}) = \tilde{r} \circ \iota(x), \\ H_2 : I \times X^n_{(k)} \to X^n_{(k)}, & H_2(0, \overline{x}) = \overline{x}, & H_2(1, \overline{x}) = \iota \circ \tilde{r}(x). \end{array}$$

Thus we have a d-homotopy equivalence $I^n_{(k)} \simeq X^n_{(k)}$. Via precomposition we get a homotopy equivalence of associated directed path spaces and hence a homotopy equivalence of trace spaces. □

We will now cover $I^n_{(k)}$ by appropriate subsets, such that the nerve lemma can be applied in a similar fashion as in [R]. Let $[1 : m]$ denote the set of integers from 1 to m.

Definition 2.2. For non-negative integers j and $i_1, i_2, \ldots, i_r \in [1 : n]$ we define the following (possibly empty) subspace of the n-cube:

$$X^{\#j}_{i_1,i_2,\ldots,\imath_r} = \{\overline{x} \in I^n | \quad x_{i_1} = x_{i_2} = \cdots = x_{i_s} = 0 \text{ and } \#\{i | x_i = 1\} \geq j\}.$$

Furthermore, $X^{\#j}$ denotes the subset of the n-cube consisting of points with at least j coordinates equal to 1 and $X_{i_1,i_2,\ldots,i_r} = X^{\#0}_{i_1,i_2,\ldots,i_r}$.

Notation 2.3. Inj_m^n denotes the set of injective maps from $[1:m]$ to $[1:n]$.

Definition 2.4. For $\alpha \in \mathrm{Inj}_{n-k}^n$ we define the subspace $\mathcal{U}_\alpha \subseteq I_{(k)}^n$ by

$$\mathcal{U}_\alpha = X^{\#(n-k)} \cup X_{\alpha(1)}^{\#(n-k-1)} \cup X_{\alpha(1),\alpha(2)}^{\#(n-k-2)} \cup \cdots \cup X_{\alpha(1),\alpha(2),\ldots,\alpha(n-k)}.$$

Proposition 2.5. *Let $\vee$ denote the coordinate-vise maximum operation on $\mathbb{R}^n$ and let $\wedge$ denote the minimum operation on the integers. For all $\alpha \in \mathrm{Inj}_{n-k}^n$ and r, s there is an inclusion*

$$X_{\alpha(1),\ldots,\alpha(r)}^{\#(n-k-r)} \vee X_{\alpha(1),\ldots,\alpha(s)}^{\#(n-k-s)} \subseteq X_{\alpha(1),\ldots,\alpha(r\wedge s)}^{\#(n-k-r\wedge s)}.$$

Thus $\mathcal{U}_\alpha$ is $\vee$-closed and so is any intersection of $\mathcal{U}_\alpha$'s.

Proof. Assume that $\overline{x} \in X_{\alpha(1),\ldots,\alpha(r)}^{\#(n-k-r)}$ and $\overline{y} \in X_{\alpha(1),\ldots,\alpha(s)}^{\#(n-k-s)}$. Then the point $\overline{x} \vee \overline{y}$ has at least $(n-k-r) \vee (n-k-s) = n-k-r\wedge s$ coordinates which are equal to 1 and $(\overline{x} \vee \overline{y})_{\alpha(1)} = \cdots = (\overline{x} \vee \overline{y})_{\alpha(r\wedge s)} = 0$. □

By Proposition 2.8 of [R] we have the following:

Corollary 2.6. *The trace space*

$$\vec{T}(\bigcap_{\alpha\in A} \mathcal{U}_\alpha)(\overline{0},\overline{1})$$

is either empty or contractible for every nonempty family of injections $A \subseteq \mathrm{Inj}_{n-k}^n$.

Theorem 2.7.

$$\vec{T}(I_{(k)}^n)(\overline{0},\overline{1}) = \bigcup_{\alpha\in\mathrm{Inj}_{n-k}^n} \vec{T}(\mathcal{U}_\alpha)(\overline{0},\overline{1})$$

Proof. For $K, L \subseteq [1:n]$ we introduce the notation

$$\mathcal{U}_K^L = \{\overline{x} \in I^n | \quad \forall i \in K : x_i = 0, \quad \forall j \in L : x_j = 1\}.$$

Let $\gamma \in \vec{P}(I_{(k)}^n)(\overline{0},\overline{1})$. Choose $t_1 > 0$ minimal such that $\exists j : \gamma_j(t_1) = 1$. Define sets

$$\begin{aligned} J_1 &= \{j \in [1:n] | \quad \gamma_j(t_1) = 1\}, \\ K_1 &= \{\ell \in [1:n] | \quad \gamma_\ell(t) = 0 \text{ for } t < t_1\}. \end{aligned}$$

We have

$$n-k \le |K_1|, \quad 1 \le |J_1| \le k, \quad K_1 \cap J_1 = \emptyset.$$

Inductively, choose $t_i > t_{i-1}$ minimal such that $\gamma_j(t_i) = 1$ for some $j \in [1:n]$ with $j \notin J_1 \cup \cdots \cup J_{i-1}$. Define sets

$$\begin{aligned} J_i &= \{j \in [1:n] | \quad \gamma_j(t_i) = 1, \quad j \notin J_1 \cup \cdots \cup J_{i-1}\}, \\ K_i &= \{\ell \in [1:n] | \quad \gamma_\ell(t) = 0 \text{ for } t < t_i\}. \end{aligned}$$

We have

$$n-k \le |K_i| + |J_1 \cup \cdots \cup J_{i-1}|, \quad 1 \le |J_i| \le k, \quad K_i \cap (J_1 \cup \cdots \cup J_i) = \emptyset.$$

Stop at $t_m > t_{m-1}$ when $\gamma_i(t) = 1$ for all $t \geq t_m$ and all $j \in [1:n]$. Note hat $m \leq n$. We now have $K_1 \supseteq K_2 \supseteq \cdots \supseteq K_m = \emptyset$ and

$$\gamma([0,t_1[) \subseteq \mathcal{U}_{K_1}^{\emptyset}, \quad \gamma([t_1,t_2[) \subseteq \mathcal{U}_{K_2}^{J_1}, \dots, \gamma([t_{i-1},t_i]) \subseteq \mathcal{U}_{K_i}^{J_1\cup\cdots\cup J_{i-1}}, \dots,$$
$$\gamma([t_{m-1},t_m]) \subseteq \mathcal{U}_{K_m}^{J_1\cup\cdots\cup J_{m-1}}, \quad \gamma([t_m,1]) \subseteq \mathcal{U}_{\emptyset}^{[1:n]}.$$

Next, we form an appropriate subsequence. Choose i_1 such that $n-k \leq |K_{i_1}|$ but $n-k > |K_{i_1+1}|$. Put $A_1 = K_{i_1}$ and $B_1 = \emptyset$. We have

$$A_1 \cap B_1 = \emptyset, \quad n-k \leq |A_1| + |B_1|.$$

Inductively, choose $i_j > i_{j-1}$ such that $K_{i_j} = K_{i_{j-1}+1}$ but $K_{i_j+1} \neq K_{i_{j-1}+1}$. Put $A_j = K_{i_j}$ and $B_j = J_1 \cup \cdots \cup J_{i_{j-1}}$. For some s we have $K_{i_{s-1}+1} = \emptyset$. Here we finish by $A_s = \emptyset$, $B_s = J_1 \cup \cdots \cup J_{i_{s-1}}$. We now have a strictly decreasing sequence $A_1 \supset A_2 \supset \cdots \supset A_s = \emptyset$ and a sequence $\emptyset = B_1 \subseteq B_2 \subseteq \cdots \subseteq B_s$ such that

$$A_i \cup B_i = \emptyset, \quad n-k \leq |A_i| + |B_i|$$

for all i. Furthermore, we have

$$\gamma([0,t_{i_1}[) \subseteq X_{A_1}^{B_1}, \dots, \gamma([t_{i_{j-1}}, t_{i_j}[) \subseteq X_{A_j}^{B_j}, \dots, \gamma([t_s,1]) \subseteq X_{A_s}^{B_s}.$$

such that

$$\gamma([0,1]) \subseteq \bigcup_{j=1}^{s} X_{A_j}^{B_j}.$$

Put $n_i = |A_i|$ and choose $\alpha \in \mathrm{Inj}_{n-k}^n$ such that

$$\{\alpha(1), \dots, \alpha(n_i)\} = A_i, \text{ for } 2 \leq i \leq s-1$$

and $\{\alpha(1), \dots, \alpha(n-k)\} \subseteq A_1$. Then $\gamma([0,1]) \subseteq \mathcal{U}_\alpha$. □

As in [R], we need a covering of $I_{(k)}^n$ by *open* subsets of I^n. Choose $0 < \epsilon < \frac{1}{3}$. Replace the conditions $x_i = 0$ and $x_j = 1$ by $x_i < \epsilon$ and $x_j > 1-\epsilon$ respectively in the above definition of the covering. Then one gets an open covering and an analogue of Proposition 2.12 in [R] holds.

3 A simplicial complex model of the trace space

Definition 3.1. Let $\mathcal{C}_k^n$ denote the poset

$$\mathcal{C}_k^n = \{A \subseteq \mathrm{Inj}_{n-k}^n | \quad A \neq \emptyset, \quad \vec{T}(\bigcap_{\alpha\in A} \mathcal{U}_\alpha)(\overline{0},\overline{1}) \neq \emptyset\}.$$

The partial order is inclusion.

For an object $A \in \mathcal{C}_k^n$ we have a simplex in $\mathbb{R}^{\mathrm{Inj}_{n-k}^n} \cong \mathbb{R}^{n!/k!}$ as follows:

$$\Delta^{|A|-1} = \{(x_\alpha)| \quad x_\alpha \geq 0 \text{ for all } \alpha, \quad \sum x_\alpha = 1, \quad x_\alpha = 0 \text{ if } \alpha \notin A\}.$$

We define functors

$$\mathcal{D} : (\mathcal{C}_k^n)^{op} \to \mathbf{Top}; \qquad A \mapsto \vec{T}(\bigcap_{\alpha \in A} \mathcal{U}_\alpha)(\overline{0}, \overline{1}),$$
$$\mathcal{E} : \mathcal{C}_k^n \to \mathbf{Top}; \qquad A \mapsto \Delta^{|A|-1}.$$

An argument similar to the one given [R], proof of Theorem 3.5, shows that the trace space of $I^n_{(k)}$ is homotopy equivalent to the colimit of $\mathcal{E}$ and to the nerve of $\mathcal{C}_k^n$. Thus, we have the following result:

Theorem 3.2. *There is a homotopy equivalence*

$$\vec{T}(I^n_{(k)})(\overline{0}, \overline{1}) \simeq \mathcal{N}(\mathcal{C}_k^n).$$

We will now give a better description of the category $\mathcal{C}_k^n$. An intersection of subsets can be described as follows:

Lemma 3.3. *Let $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathrm{Inj}^n_{n-k}$. Then one has*

$$\bigcap_{j=1}^r \mathcal{U}_{\alpha_j} = \bigcup_{i=0}^{n-k} X^{\#(n-k-i)}_{\alpha_1(1),\dots,\alpha_1(i),\alpha_2(1),\dots,\alpha_2(i),\dots,\alpha_r(1),\dots,\alpha_r(i)}.$$

Proof. We insert the expression defining $\mathcal{U}_{\alpha_j}$ and obtain

$$\begin{aligned}\bigcap_{j=1}^r \mathcal{U}_{\alpha_j} &= \bigcap_{j=1}^r (X^{\#(n-k)} \cup X^{\#(n-k-1)}_{\alpha_j(1)} \cup X^{\#(n-k-2)}_{\alpha_j(1),\alpha_j(2)} \cup \dots \cup X_{\alpha_j(1),\alpha_j(2),\dots,\alpha_j(n-k)}) \\ &= X^{\#(n-k)} \cup \bigcap_{j=1}^r (X^{\#(n-k-1)}_{\alpha_j(1)} \cup X^{\#(n-k-2)}_{\alpha_j(1),\alpha_j(2)} \cup \dots \cup X_{\alpha_j(1),\alpha_j(2),\dots,\alpha_j(n-k)})\end{aligned}$$

since the term $X^{\#(n-k)}$ appears in every factor and when we intersect it by another summand, we get a subset of $X^{\#(n-k)}$. By a similar argument, we can rewrite the above as

$$X^{\#(n-k)} \cup X^{\#(n-k-1)}_{\alpha_1(1),\alpha_2(1),\dots,\alpha_r(1)} \cup \bigcap_{j=1}^r (X^{\#(n-k-2)}_{\alpha_j(1),\alpha_j(2)} \cup \dots \cup X_{\alpha_j(1),\alpha_j(2),\dots,\alpha_j(n-k)}).$$

We continue this way, and get the desired result. □

We can now formulate a useful criterion for whether the trace space of an intersection of subsets is empty or not.

Corollary 3.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathrm{Inj}^n_{n-k}$. Then one has*

$$\vec{T}(\bigcap_{j=1}^r \mathcal{U}_{\alpha_j})(\overline{0}, \overline{1}) \neq \emptyset \iff$$
$$\forall i : \#\{\alpha_1(1), \dots, \alpha_1(i), \alpha_2(1), \dots, \alpha_2(i), \dots, \alpha_r(1), \dots, \alpha_r(i)\} \leq i + k - 1.$$

Proof. The end point $\overline{1}$ lies in $X^{\#(n-k)}$ and not in any other summand. The starting point $\overline{0}$ lies only in $X_{\alpha_1(1),\dots,\alpha_1(n-k),\dots,\alpha_r(1),\dots,\alpha_r(n-k)}$. There exists a directed path from $\overline{0}$ which moves further ahead precisely when

$$\#\{\alpha_1(1),\dots,\alpha_1(n-k),\dots,\alpha_r(1),\dots,\alpha_r(n-k)\} \leq n-1.$$

Assuming that, we can arrive at a point in $X^{\#1}_{\alpha_1(1),\dots,\alpha_1(n-k-1),\dots,\alpha_r(1),\dots,\alpha_r(n-k-1)}$. There exists a directed path further ahead from this point precisely when

$$1+\#\{\alpha_1(1),\dots,\alpha_1(n-k-1),\dots,\alpha_r(1),\dots,\alpha_r(n-k-1)\} \leq n-1.$$

Assuming that, we can get to a point in $X^{\#2}_{\alpha_1(1),\dots,\alpha_1(n-k-2),\dots,\alpha_r(1),\dots,\alpha_r(n-k-2)}$ and so on. If we have arrived at a point in $X^{\#(n-k)}$, we can always move further to the end point $\overline{1}$. □

Definition 3.5. For $A \subseteq \mathrm{Inj}^n_{n-k}$ we let

$$\mathrm{Im}_i(A) = \bigcup_{\alpha \in A} \{\alpha(1), \alpha(2), \dots, \alpha(i)\}.$$

Theorem 3.6. *The poset $\mathcal{C}^n_k$ has the following purely combinatorial description:*

$$\mathcal{C}^n_k = \{A \subseteq \mathrm{Inj}^n_{n-k} | \quad \forall i : i \leq |\,\mathrm{Im}_i(A)| \leq i+k-1\}.$$

4 The flag-category model

Definition 4.1. The *flag category* is the poset

$$\mathcal{F}^n_k = \{(F_1 \subseteq F_2 \subseteq \dots \subseteq F_{n-k})| \quad F_{n-k} \subseteq [1:n], \quad \forall i : i \leq |F_i| \leq i+k-1\}.$$

The partial order is given by $\underline{F} \subseteq \underline{G} \Leftrightarrow \forall i : F_i \subseteq G_i$.

Note that there is a forgetful functor

$$U : \mathcal{C}^n_k \to \mathcal{F}^n_k; \quad A \mapsto (\mathrm{Im}_1(A), \mathrm{Im}_2(A), \dots, \mathrm{Im}_{n-k}(A)).$$

We now define a functor, which turns out to be the left adjoint of U.

Definition 4.2. The functor $V : \mathcal{F}^n_k \to \mathcal{C}^n_k$ is defined by

$$V(\underline{F}) = \{\alpha \in \mathrm{Inj}^n_{n-k} | \quad U(\{\alpha\}) \subseteq \underline{F}\}.$$

Note that $V(\underline{F})$ is a final among the objects A with property $U(A) = \underline{F}$.

Theorem 4.3. *$V : \mathcal{F}^n_k \to \mathcal{C}^n_k$ is the left adjoint of the forgetful functor $U : \mathcal{C}^n_k \to \mathcal{F}^n_k$. Thus there is a homotopy equivalence*

$$\mathcal{N}(\mathcal{C}^n_k) \xrightarrow[\simeq]{\mathcal{N}(U)} \mathcal{N}(\mathcal{F}^n_k), \qquad \mathcal{N}(\mathcal{C}^n_k) \xleftarrow[\simeq]{\mathcal{N}(V)} \mathcal{N}(\mathcal{F}^n_k).$$

Proof. The composite UV is the identity on $\mathcal{F}_k^n$, so the counit of the adjunction is simply the identity

$$UV \xrightarrow{\mathrm{Id}} \mathrm{Id}_{\mathcal{F}_k^n}.$$

By definition of V we have an inclusion $A \hookrightarrow VU(A)$ for all objects $A \in \mathcal{C}_k^n$. This defines the unit of the adjunction

$$\mathrm{Id}_{\mathcal{C}_k^n} \longrightarrow VU.$$

A functor induces a map of nerves and a natural transformation of functors induces a homotopy between such maps. Thus an adjoint pair of functors induces a homotopy equivalence. □

By combining Theorems 3.2 and 4.3 we get our main result:

Theorem 4.4. *There is a homotopy equivalence*

$$\vec{T}(I^n_{(k)})(\overline{0},\overline{1}) \simeq \mathcal{N}(\mathcal{F}_k^n).$$

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